The Kármán vortex street phenomenon

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Introduction

This note is for the doctoral students in the applied finite element analysis class that covered various physics phenomena, including fluid dynamics. Surprisingly, despite having engineering, or even physics, undergraduate degrees, the students were unaware of the very important Kármán's vortex phenomenon. The following is a brief overview of the subject.

1. The vorticity equation and its solution

The velocity vector of a moving fluid particle is a function of its location and time:

$\vec{v} = \vec{v}(x, y, z, t) = (u, v, w)$.

The Navier-Stokes equation describing its equilibrium is described by

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla p + \rho \nabla (gh) + \mu \Delta \vec{v}$$

where ρ is the uniform density of the fluid, $p(x, y, z)$ is the scalar pressure, $g, h(x, y, z)$ are the acceleration and scalar potential of gravity, and μ is the coefficient of friction. The vorticity of the flow is defined by the operation

$\vec{\omega} = \nabla \times \vec{v}$.

Applying the operation to the Navier-Stokes equation we obtain

$$\rho \frac{\partial \vec{\omega}}{\partial t} + \rho\left((\vec{v} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{v}\right) = -\nabla \times \nabla p + \rho \nabla \times \nabla gh + \mu \Delta \vec{\omega} .$$

Since $\nabla \times \nabla$ operation on a scalar is zero, the pressure and gravitational terms on the right-hand side drop out.

Dividing by the density and introducing the kinematic viscosity $\nu = \frac{\mu}{\rho}$, we obtain the general three-dimensional vorticity equation

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{v} = \nu \, \nabla^2 \vec{\omega}.$$

Let us now restrict our work to a horizontal two-dimensional flow.

In this case $\vec{\omega} = (0, 0, \omega)$ where the scalar ω represents the vertical component of the vorticity vector. Furthermore, in this case $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)$, therefore $(\vec{\omega} \cdot \nabla) = 0$ results in elimination of the third term on the left-hand side. The second term on the left-hand side becomes

$$(\vec{v} \cdot \nabla)\vec{\omega} = u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y}.$$

Hence, the two-dimensional horizontal vorticity equation is of the form

$$\frac{\partial \omega}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y})\omega = \nu \, \nabla^2 \omega.$$

By introducing a scalar stream function $\varphi(x, y)$, such that

$$u = -\frac{\partial \varphi}{\partial y}; \; v = \frac{\partial \varphi}{\partial x},$$

the vorticity differential equation for this scenario is

$$\frac{\partial \omega}{\partial t} - \frac{\partial \varphi}{\partial y}\frac{\partial \omega}{\partial x} + \frac{\partial \varphi}{\partial x}\frac{\partial \omega}{\partial y} = \nu \, \nabla^2 \omega.$$

The stream function is constant on a streamline of the flow; hence it will enable the visualization of the vorticity pattern of the flow. By the definition of the stream function, we can write

$$\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right) = (v, -u) \text{ and then } \nabla \cdot \nabla \varphi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

The vorticity of the two-dimensional stream function is

$$\vec{\omega} = \nabla \times \vec{v} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\vec{k}$$, hence the following relationship holds: $\nabla^2 \varphi = \omega$.

This will be the auxiliary equation used in the solution. A practical numerical solution may be accomplished by a simultaneous discretization in time and in space.

The temporal discretization with equidistant time steps is

$$0..,t-1,t,t+1,..,t_{\max};\ t-(t-1)=\Delta t=(t+1)-t\,.$$

Spatial discretization will be a two-dimensional finite difference scheme shown on Figure 1.

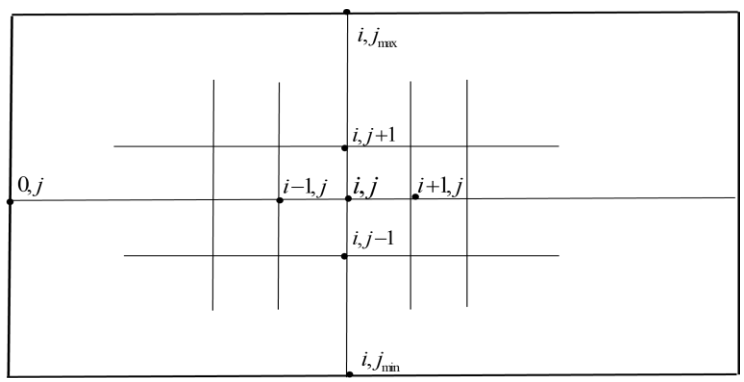

Figure 1.

Hence the discretized vorticity equation becomes:

$$\frac{\omega_{i,j}^{t+1}-\omega_{i,j}^{t-1}}{2\Delta t}-\frac{\left(\varphi_{i,j+1}^{t}-\varphi_{i,j-1}^{t}\right)\left(\omega_{i+1,j}^{t}-\omega_{i-1,j}^{t}\right)}{\left(2\Delta x\right)^2}+\frac{\left(\varphi_{i+1,j}^{t}-\varphi_{i-1,j}^{t}\right)\left(\omega_{i,j+1}^{t}-\omega_{i,j-1}^{t}\right)}{\left(2\Delta x\right)^2}=\nu\frac{\nabla^2\omega_{i,j}^{t+1}-\nabla^2\omega_{i,j}^{t-1}}{(\Delta x)^2}\,.$$

Multiplying and reordering produce the equation of the numerical procedure

$$\nabla^2\omega_{i,j}^{t+1}-\frac{(\Delta x)^2}{\nu 2\Delta t}\omega_{i,j}^{t+1}=$$

$$=\nabla^2\omega_{i,j}^{t-1}-\frac{(\Delta x)^2}{\nu 2\Delta t}\omega_{i,j}^{t-1}+\frac{1}{4\nu}\left[-\left(\varphi_{i,j+1}^{t}-\varphi_{i,j-1}^{t}\right)\left(\omega_{i+1,j}^{t}-\omega_{i-1,j}^{t}\right)+\left(\varphi_{i+1,j}^{t}-\varphi_{i-1,j}^{t}\right)\left(\omega_{i,j+1}^{t}-\omega_{i,j-1}^{t}\right)\right]$$

The initial conditions at the inflow wall of the two-dimensional flow domain are

$$\omega_{0,j}=0;\ u_{0,j}=U;\ v_{0,j}=0;\ \varphi_{0,j}=const\,.$$

The boundary conditions on the walls represent a smooth fluid slip along the surface:

$$\omega_{i,j\max}=\omega_{i,j\min}=0;\ \varphi_{i,j}=const\,.$$

The numerical solution is executed by stepping through the time and for all grid locations, solving the discretized vorticity equation for the next vorticity value $\omega_{i,j}^{t+1}$ and the auxiliary equation for the stream function value $\varphi_{i,j}^{t+1}$, while adhering to the initial and boundary conditions. The solution also accommodates a body placed in the flow, a topic of our main interest. The shape of the immersed object of course heavily influences the constant velocity streamlines and corresponding vorticity values in the analyzed flow domain.

2. The Kármán vortex street phenomenon

The phenomenon, originally recognized by and now universally attributed to Theodor von Kármán, occurs when an object, for example a cylinder, is placed in a flow. Let us consider a flow between two walls, as shown in Figure 2., and assume that the object is in some distance from the upper and lower boundaries, as well as far away from the initial and final boundaries.

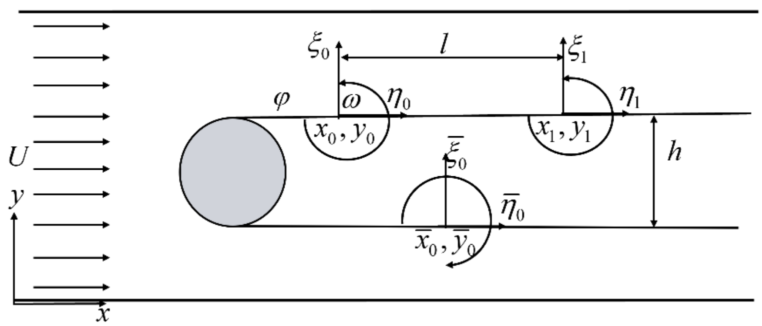

Figure 2.

The initially uniform and not turbulent horizontal flow will be separated by the cylinder. Due to the lower pressure beyond the cylinder, the flow is faster and creates vortices as shown in the image. Kármán described and analyzed this periodic phenomenon, now called the Kármán vortex street, in a pair of famous pioneering papers in 1911 and 1912, shown in the Reference section. Figure 3 is in fact from Kármán's first paper on the subject.

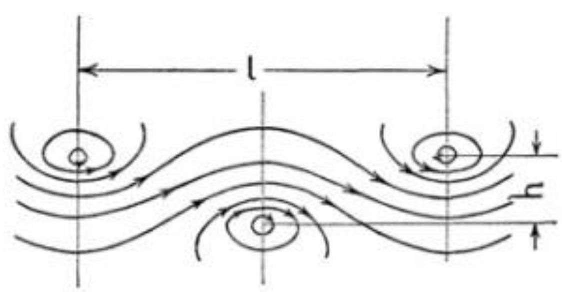

Figure 3.

Kármán stated that the only vortex pattern that is stable in time is an alternating set of vortices in two rows. He also established the values of geometrical spacing and the velocity of the vortices. Furthermore, he computed the force generated on the object by the flow and the phenomenon, important in engineering practice.

Kármán described the velocity components of the p-th vortex as

$$\frac{dx_p}{dt} = \frac{\omega}{2\pi}\sum_{q\neq p}\frac{y_p - y_q}{(x_p - x_q)^2 + (y_p - y_q)^2};\ \frac{dy_p}{dt} = -\frac{\omega}{2\pi}\sum_{q\neq p}\frac{x_p - x_q}{(x_p - x_q)^2 + (y_p - y_q)^2}.$$

Here the q index is for all the other vortices. He then introduced local coordinates for the vortices (shown on Figure 2) as

$$x_p = p\cdot l + \xi_p;\ y_p = \eta_p,$$

where l is the spacing between the vortices on one side. The velocity components in the local coordinates are

$$\frac{2\pi}{\omega}\frac{d\xi_p}{dt} = \sum_{q\neq p}\frac{\eta_p - \eta_q}{(p-q)^2 l^2};\ \frac{2\pi}{\omega}\frac{d\eta_p}{dt} = \sum_{q\neq p}\frac{\xi_p - \zeta_q}{(p-q)^2 l^2}.$$

The following differentiations produce the relationships

$$\frac{\partial}{\partial\xi_q}\frac{d\xi_p}{dt} = -\frac{\partial}{\partial\eta_q}\frac{d\eta_p}{dt} = \alpha_{pq};\ \frac{\partial}{\partial\eta_q}\frac{d\xi_p}{dt} = \frac{\partial}{\partial\xi_q}\frac{d\eta_p}{dt} = \beta_{pq}.$$

Substituting results in

$$\frac{2\pi}{\omega}\frac{d\xi_p}{dt} = \sum_{q\neq p}\left(\alpha_{pq}\xi_q + \beta_{pq}\eta_q\right);\ \frac{2\pi}{\omega}\frac{d\eta_p}{dt} = \sum_{q\neq p}\left(\beta_{pq}\xi_q - \alpha_{pq}\eta_q\right).$$

Focusing on the first two vortices and assuming that the solutions are all of the form

$$\xi_1, \eta_1, \xi_2, \eta_2 = e^{\frac{\omega}{2\pi}\lambda t},$$

The simultaneous solution for the velocity terms is presented by the characteristic equation of

$$\begin{bmatrix} \alpha_{11}-\lambda & \beta_{11} & \alpha_{12} & \beta_{12} \\ \beta_{11} & \alpha_{11}-\lambda & \beta_{12} & -\alpha_{12} \\ \alpha_{21} & \beta_{21} & \alpha_{22}-\lambda & \beta_{22} \\ \beta_{21} & -\alpha_{21} & \beta_{22} & -\alpha_{22}-\lambda \end{bmatrix}\begin{bmatrix}\xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2\end{bmatrix} = 0.$$

Since from the definition

$$\alpha_{12} = -\alpha_{21};\ \beta_{12} = -\beta_{21},$$

the skew-symmetric determinant is easy to obtain. There are two distinct solutions, one assuming a symmetric positioning of the first two vortices, while the other one is unsymmetric. Kármán proved that the symmetric solution is unstable, however, the unsymmetric one is stable leading to a continuing vortex generation and propagation, also called vortex shedding.

The details (of translating the infinite series into hyperbolic functions) are omitted, only Kármán's conclusions are shown. For the symmetric, unstable scenario the condition

$$\cosh\frac{\pi h}{l}=\sqrt{3}\rightarrow\frac{h}{l}\cong 0.359,$$

holds, while the condition of the stable vortex solution is

$$\cosh\frac{\pi h}{l}=\sqrt{2}\rightarrow\frac{h}{l}\cong 0.283.$$

The h is the distance between the vortex rows that is initially the dimension of the object perpendicular to the flow (for example the diameter of the cylinder) and l is the spacing of the vortices on a side as was show earlier. Kármán also computed the x-directional velocity of the vortices as

$$u=\frac{\omega}{2l}\tan\left(\frac{\pi h}{l}\right).$$

Finally, his result on the force exerted on the object (or the drag force on the object if it is moving in stationary fluid), is

$$W=\rho\omega(U-u)\frac{h}{l}.$$

The most important engineering consequence of the vortex shedding is the induced vibration on the structure in the flow. The frequency of this vibration is

$$f=St\frac{U}{h},$$

where St is the Strouhal number usually between 0.18 and 0.22. The interesting historical background is that the Czech physicist was measuring the acoustic frequency of the noise generated by the vibrations of telegraph wires.

He found that it was proportional to the ratio of the wind speed and the diameter of the wire. He found that the constant of proportionality was depended upon the speed and the higher speeds produced the higher value.

This, then not yet understood phenomenon was explained by Kármán's work that essentially created a very important connection between acoustic and mechanical vibrations.

The infamous collapse of the Tacoma Bridge in Washington State happened because the vortex shedding frequency was very close to one of the natural frequencies of the bridge structure, resulting in a resonance catastrophe. The bridge is shown on Figure 4 just before its collapse.

Figure 4.

There are natural manifestations also, like the ocean vortex street generated by the wind flowing around the San Juan Fernandez Islands off the Chilean coast shown on Figure 5.

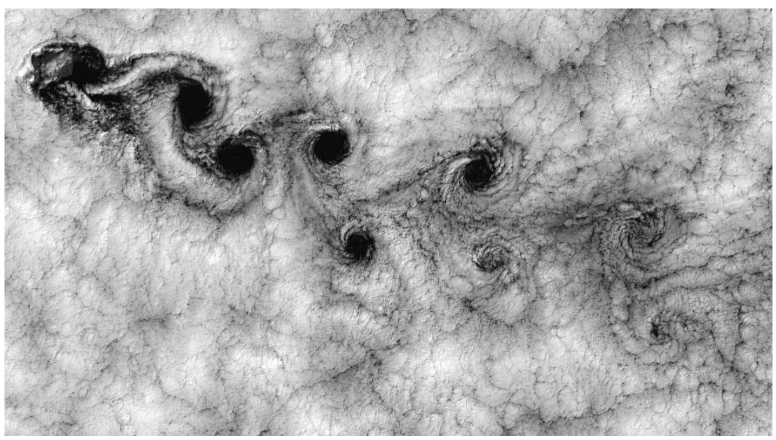

Figure 5.

Finally, while the above was focused on a 2D flow scenario, the phenomenon also exists in 3D, for example following airplane wings, and solved by commercial numerical software tools.

Reference

Kármán, Theodor von; Uber den Mechanism des Wiederstandes den ein bewegter Korper in einer Flussigkeit erfahrt.

Gottingen Nachrichten; Mathematische-Physikalische Klasse, 1911 and in Gottingen Physics Zeitung in 1912.